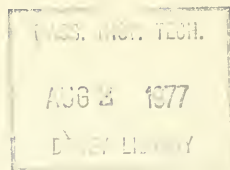


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MULTIPLE CRITERIA PUBLIC INVESTMENT DECISION MAKING

BY MIXED INTEGER PROGRAMMING*

Jeremy F. Shapiro

WP 788-75

May 1975

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Jeremy F. Shapiro

1. INTRODUCTION

Multicriterion optimization has long been an integral part of quantitative models for public planning. The models which have received the most attention consist of concave utility functions to be maximized (in a vector sense) subject to convex constraints. The convex structure of such models permits the development of complete theories characterizing efficient solutions, and well-behaved algorithms for finding these solutions.

In this paper, we consider a different class of public planning problems for which the assumption of convex constraints, or non-decreasing marginal costs of production, is not valid. There are at least three circumstances when this is the case. First, there can be public investment alternatives such as dam or power plant construction involving large fixed costs. Second, there can be important returns to scale in construction and production costs. Finally, public investment decision problems involving new technologies can have associated diffusion and learning curves which are S-shaped; that is, increasing marginal costs are experienced at first, followed by a consolidation phase when marginal costs are decreasing. All of these phenomena can be described by mixed integer programming models. There is a growing literature on decision models of this type; for example, see Bhatia [1975], Dorfman and Jacoby [1972], Goreux and Manne [1973], Kendrick and Stoutessdijk [1975], Kuhner and Harrington [1974], Marks [1974], Westphal [1971].

The main concern of this paper is the development of procedures for multicriteria optimization of integer programming problems. The specific model we will consider is the vector maximization problem

$$\begin{aligned} \text{Max} \quad & Cx \\ \text{s.t.} \quad & Ax \leq b \\ & x_j = 0 \text{ or } 1 \end{aligned} \tag{1}$$

where C is a $P \times N$ matrix, A is an $M \times N$ matrix, b is an $M \times 1$ vector, and the coefficients of A and b are integer. The columns a^j of A correspond to projects extending over time to be selected or rejected by P interested parties on the basis of the $P \times 1$ vector c^j measuring its value to these parties. The component c_p^j may equal the present net value of project j computed with a discount rate specific to the p th party. The constraints in (1) can consist of capital budget constraints for each of a number of time periods, other resource constraints, logical constraints

imposing a unique starting period for a project, precedence constraints on subsets of projects, and so on. For future reference, we define the set

$$F = \{x | Ax \leq b, x_j = 0 \text{ or } 1\}; \quad (2)$$

the set is finite, say $F = \{x^k\}_{k=1}^K$.

Different and more general multicriterion optimization problems than (1) involving indivisible investment alternatives are clearly possible. One can envision mixed integer programming models involving continuous decision variables as well as fixed charge variables. In addition, the value of a feasible solution x to the p th party may be measured by a utility function u_p . Such extensions to our basic multicriterion IP model and algorithms for finding efficient solutions to it are possible and will be discussed in the conclusions section of this paper.

As with convex vector maximization problems, there probably is not a vector maximum solution to problem (1), and we are led to a study of efficient solutions. Specifically, a solution $\tilde{x}$ satisfying the constraints of problem (1) is said to be efficient if there does not exist a feasible solution x satisfying the P inequalities.

$$Cx \geq C\tilde{x}$$

with strict inequality for some component. The characterization of efficient solutions of IP problem (1) is not as simple as it is for linear or convex multicriterion problems, and moreover, the calculation of these solutions is radically different.

The plan of this paper is as follows. Section 2 discusses mathematical properties of efficient solutions. The following section contains a brief review of IP duality theory which provides the constructive means for implementing some of the results of section 2. Section 4 contains a description of how IP duality theory is used to compute efficient solutions. Section 5 contains sensitivity analysis for computing the convex region in which a given efficient solution is optimal in a related IP problem. Some concluding remarks and areas for future research are discussed briefly in section 6.

2. MATHEMATICAL PROPERTIES OF EFFICIENT SOLUTIONS

The usual practice in generating efficient solutions to a multicriterion mathematical programming problem is to give positive weights to each of the P objective functions and combine them (for example, see Kuhn and Tucker [1951], or Karlin [1959]). To this end, let

$$S = \left\{ \lambda \in R^P \mid \sum_{p=1}^P \lambda_p = 1, \lambda_p \geq 0 \right\}$$

and let

$$\text{int } S = \{ \lambda \mid \lambda \in S \text{ and } \lambda_p > 0 \text{ for all } p \}.$$

For $\lambda \in \text{int } S$, define the IP problem

$$\begin{aligned} v(\lambda) &= \lambda Cx \\ \text{s.t. } x &\in F \end{aligned} \quad (3)$$

Theorem 1: For $\lambda \in \text{int } S$, any maximal solution to (3) is efficient.

Proof: Let x denote a maximal solution to (3) and suppose it is not efficient; that is, there exists a $y \in F$ such that $Cy \geq Cx$ with strict inequality in some component. Since $\lambda \in \text{int } S$, this clearly implies $\sum_{p=1}^p \lambda_p (Cy)_p > \sum_{p=1}^p \lambda_p (Cx)_p$ contradicting the optimality of x in (3). ||

As is well known, the difficulty with the characterization of Theorem 1 of efficient solutions is that it is a sufficient but not necessary condition for a solution to be efficient. The following example, due to Gabriel Bitran, illustrates this point. The IP problem is

$$\max \begin{bmatrix} 1, 10, 9 \\ 10, 1, 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (4)$$

$$\begin{aligned} \text{s.t. } x_1 + x_2 + x_3 &\leq 2 \\ x_j &= 0 \text{ or } 1, j = 1, 2, 3 \end{aligned}$$

There are seven feasible solutions to this problem, and three efficient solutions $x^1 = (1, 1, 0)$, $x^2 = (1, 0, 1)$, $x^3 = (0, 1, 1)$. Let λ be the weight on the top objective function in (4), and $1-\lambda$ be the weight the bottom objective function. It is easy to verify that x^2 is optimal in the IP problem (3) derived from (4) for $\lambda \in [0, 1/2]$, and x^3 is optimal for $\lambda \in [1/2, 1]$, but x^1 is not optimal for any $\lambda \in S$.

This anomaly of efficient IP solutions needs further study and will not be reconciled here. Instead, we focus on parametric computation of $v(\lambda)$ for $\lambda \in \text{int } S$. It can easily be shown that $v(\lambda)$ is a piecewise linear convex function on S . Given any $x^k \in F$, $v(\lambda) = \lambda Cx^k$ for λ satisfying

$$\begin{aligned} \lambda Cx^k &\geq \lambda Cx^k, \quad k = 1, \dots, F \\ \sum_{p=1}^p \lambda_p &= 1 \\ \lambda_p &\geq 0. \end{aligned} \quad (5)$$

Let $S(x^k)$ denote the region of S defined in (5); clearly x^k is efficient by Theorem 1 if $S(x^k) \cap \text{int } S \neq \emptyset$.

Parametric variation of $\lambda \in \text{int } S$ for multicriterion LP is possible by sensitivity analysis which relies in turn on LP duality theory. The IP problem (3) can be expressed as the LP problem

$$\begin{aligned} v(\lambda) &= \max \lambda Cx \\ \text{s.t. } x &\in [F] \end{aligned} \quad (6)$$

where "[]" denotes convex hull. This last statement is true because all extreme points of $[F]$ are solutions in F . In fact, the zero-one structure of problem (1) implies all solutions in F are extreme points of $[F]$; that is, x is an extreme point of $[F]$ if and only if $x \in F$. The difficulty with this apparent reduction of problem (3) to a more manageable optimization problem is that $[F]$ is generally impossible to characterize in any real practical sense. The IP duality theory reviewed in the next section, and used in subsequent sections, can be interpreted as approximating $[F]$ in a neighborhood of an optimal IP solution. As a result it provides insights on how to perform sensitivity analysis as λ varies in $\text{int } S$ in problem (3).

3. REVIEW OF IP DUALITY THEORY

In this section we review briefly the IP duality theory developed in Shapiro [1971], Fisher and Shapiro [1974], Bell [1973a][1973b], Bell and Shapiro [1975]. We consider the zero-one IP problem

$$\begin{aligned} v &= \max cx \\ \text{s.t. } Ax + Is &= b \\ x_j &= 0 \text{ or } 1 \\ s_i &= 0, 1, 2, \dots, U_i \end{aligned} \quad (7)$$

which is identical to problem (1) except c is $1 \times N$. The integer U_i is an upper bound on the value of the slack s_i . The slacks in (7) are integer variables because A and b are integer.

A dual problem to (7) is constructed by reformulating it as follows. Let G be any finite abelian group with the representation

$$G = Z_{q_1} \oplus Z_{q_2} \oplus \dots \oplus Z_{q_r}$$

where the positive integers q_i satisfy $q_i \geq 2$, $q_i | q_{i+1}$, $i=1, \dots, r-1$, and Z_{q_i} is the cyclic group of order q_i . Let $|G|$ denote the order of G ; clearly $|G| = \prod_{i=1}^r q_i$. Let $e_1, \dots, e_M$ be any elements of this group and for any M -vector

f, define the element $\phi(f) = \sigma \in G$ by

$$\phi(f) = \sum_{i=1}^M f_i e_i.$$

It can easily be shown that (7) is equivalent to (has the same feasible region as)

$$v = \max cx \quad (8a)$$

$$\text{s.t. } Ax + Is = b \quad (8b)$$

$$\sum_{j=1}^N a_{ij} x_j + \sum_{i=1}^M e_i s_i = \beta \quad (8c)$$

$$x_j = 0 \text{ or } 1$$

$$s_i = 0, 1, 2, \dots, M_i \quad (8d)$$

where $a_{ij} = \phi(a_j)$ and $\beta = \phi(b)$. The group equations (8c) are a system of r congruences and they can be viewed as an aggregation of the linear system $Ax + Is = b$. Hence the equivalence of (7) and (8). For future reference, let Y be the set of (x, s) solutions satisfying (8c) and (8d).

The IP dual problem induced by G is constructed by dualizing with respect to the constraints $Ax + Is = b$. Specifically, for each u , define the Lagrangean problem

$$L(u) = ub + \max_{(x,s) \in Y} \{(c-uA)x - us\}.$$

It is well known and easily shown that the function $L(u)$ is convex, continuous and an upper bound on v . The IP dual problem is to find the least upper bound

$$w = \min L(u)$$

$$\text{s.t. } u \in R^M \quad (9)$$

If $w = -\infty$, then the IP problem (7) is infeasible.

The desired relation of the IP dual problem (9) to the primal IP problem (7) is summarized by the following:

OPTIMALITY CONDITIONS: The pair of solutions $(x^*, s^*) \in Y$ and $u^* \in R^M$ is said to satisfy the optimality conditions if

$$(i) L(u^*) = u^*b + (c-u^*A)x^* - u^*s$$

$$(ii) Ax^* + Is^* = b.$$

It can easily be shown that such a pair is optimal in the respective primal and dual problems. For a given IP dual problem, there is no guarantee that the optimality conditions can be established, but attention can be restricted to optimal dual solutions for which we try to find a complementary optimal primal solution. If the dual IP problem cannot be used to solve the primal problem, $v < w$ and we say there is a duality gap.

Solution of the IP problem (7) by dual methods is constructively achieved by generating a finite sequence of groups $\{G^k\}_{k=0}^K$, sets $\{Y^k\}_{k=0}^K$, and IP dual problems analogous to (9) with minimal objective function value w . The group $G^0 = Z_1$, $Y^0 = \{(x, z) | x_j = 0 \text{ or } 1, s_i = 0, 1, 2, \dots, U_i\}$ and the corresponding dual problem can be shown to be the linear programming relaxation of (7). The groups here have the property that G^k is a subgroup of G^{k+1} , implying directly that $Y^{k+1} \subseteq Y^k$ and therefore that $v \leq w^{k+1} \leq w^k$. Sometimes we will refer to G^{k+1} as a supergroup of G^k .

The critical step in this approach to solving the IP problem (7) is that if an optimal solution to the k th dual does not yield an optimal integer solution, then we are able to construct the supergroup G^{k+1} so that $Y^{k+1} \subset Y^k$. Moreover, the construction eliminates the infeasible IP solutions $(x, s) \in Y^k$ which are used in combination by the IP dual problem to produce a fractional solution to the optimality conditions. Since the set of feasible solutions to (7) is finite, the process must converge in a finite number of IP dual problem constructions to an IP dual problem yielding an optimal solution to (7) by the optimality conditions.

The IP dual problem (9) is actually a large scale linear programming problem. Let $Y = \{x^t, s^t\}_{t=1}^T$ be an enumeration of Y . The IP formulation of (9) is

$$\begin{aligned} v &= \min v \\ v &\geq ub + (c-ua)x^t - us^t \\ t &= 1, \dots, T \end{aligned} \quad (10)$$

The linear programming dual to (10) is

$$\begin{aligned} v &= \max \sum_{t=1}^T (cx^t) \omega_t \\ \text{s.t. } \sum_{t=1}^T (Ax^t + Is^t) \omega_t &= b \\ \sum_{t=1}^T \omega_t &= 1 \\ \omega_t &\geq 0. \end{aligned} \quad (11)$$

The number of rows T in (10), or columns in (11), is enormous. The solution methods given in Fisher and Shapiro [1974], generate columns as needed by descent algorithms for solving (10) and (11) as a primal dual pair. The columns are generated by solving the Lagrangean problem which can be solved in a matter of a few seconds for values of $|C|$ up to 3000; see Glover [1969] or Gorry, Northup and Shapiro [1973].

The formulation (11) has a convex analysis interpretation. Specifically, the feasible region in (11) corresponds to

$$\{(x, s) | Ax + Is = b, 0 \leq x_j \leq 1, 0 \leq s_i \leq U_i\} \cap [Y]$$

where the left hand set is the feasible region of the LP relaxation of (7). Thus, in effect, the dual approach approximates the convex hull of the set of feasible integer points $[F]$ by the intersection of the LP feasible region with the polyhedron $[Y]$. When the IP dual problem (9) solves the IP Problem (7), then $[Y]$ has cut away enough of the LP feasible region to approximate the convex hull of feasible integer solutions in a neighborhood of an optimal IP solution.

4. CALCULATION OF EFFICIENT SOLUTIONS

Solution methods for the multicriterion IP problem (3) for a fixed value of $\lambda^0 \in \text{int } S$ are basically methods for solving arbitrary IP problems. The generation of efficient solutions as λ varies in $\text{int } S$, however, facilitates the use of the IP dual methods outlined in the previous section in computing $v(\lambda^0)$ and an optimal solution with this value.

The IP problem we wish to solve at λ^0 is (rewriting problem (3)).

$$\begin{aligned} v(\lambda^0) &= \max \lambda^0 Cx \\ \text{s.t. } Ax + Is &= b \\ x_j &= 0 \text{ or } 1; s_i = 0, 1, \dots, U_i \end{aligned} \tag{12}$$

From previous calculations, we have the efficient solutions $x^1, \dots, x^Q$ and we compute

$$\hat{v}(\lambda^0) = \text{maximum}_{q=1, \dots, Q} \lambda^0 Cx^q;$$

$\hat{v}(\lambda^0)$ is an initial lower bound on the value of an optimal solution to (12).

As discussed in the previous section, a dual problem to (12) is constructed from an abelian group G inducing a set Y which includes all feasible solutions to (12), but which eliminates many infeasible solutions. For $u \in R^M$, the Lagrangean problem is

$$L(u|\lambda^0) = ub + \max_{(x,s) \in Y} \{(\lambda^0 C - uA)x - us\} \quad (13)$$

where

$$Y = \{(x,s) \mid \sum_{j=1}^N a_j x_j + \sum_{i=1}^M e_i s_i = b, x_j = 0 \text{ or } 1, s_i = 0, 1, \dots, U_i\},$$

and the IP dual problem is

$$\begin{aligned} w(\lambda^0) &= \min L(u|\lambda^0) \\ \text{s.t. } u &\in R^M. \end{aligned} \quad (14)$$

For future reference, the maximization in (13) is written as $ub + G(\beta|u, \lambda^0)$; the algorithms given by Shapiro [1968], Glover [1969], Corry, Northup and Shapiro [1973], compute $G(\sigma|u, \lambda^0)$ for all $\sigma \in G$ as a natural by-product of computing $G(\beta|u, \lambda^0)$.

For any $\lambda \in S$, we define the set

$$J(\lambda) = \{j \mid x_j = 0 \text{ in (12) without loss of optimality}\}. \quad (15)$$

The IP dual methods at $\lambda = \lambda^0$ can be viewed as building up the set $J(\lambda^0)$ until an optimal solution is found by the optimality conditions. Alternatively, the IP dual methods can be abandoned at any point and branch and bound methods can be performed using only projects j for $j \in J^C(\lambda^0)$.

Theorem 2: Without loss of optimality, the project k can be eliminated from consideration in (12) (i.e. $k \in J(\lambda^0)$) if, for any $u \in R^M$,

$$\lambda^0 c^k + u(b - a^k) + G(\beta - \alpha_k | u, \lambda^0) \leq \bar{v}(\lambda^0) \quad (16)$$

Proof: Let $v(\lambda^0 | x_k = 1)$ denote the value of (12) with $x_k = 1$. We have

$$v(\lambda^0 | x_k = 1) = \lambda^0 c^k + \max_{\substack{j=1 \\ j \neq k}}^N c^j x_j$$

$$\text{s.t. } \sum_{\substack{j=1 \\ j \neq k}}^N a^j x_j + 1s = b - a^k$$

$$x_j = 0 \text{ or } 1, \quad j \neq k$$

$$s_i = 0, 1, \dots, U_i$$

Dualizing, we obtain the following upper bound on $v(\lambda^0 | x_k = 1)$

$$\begin{aligned} & \lambda^0 c^k + u(b - a^k) + \max_{\substack{j=1 \\ j \neq k}}^N (\lambda^0 c^j - u a^j) x_j \\ \text{s.t. } & \sum_{\substack{j=1 \\ j \neq k}}^N a_j x_j + \sum_{i=1}^M f_i s_i = b - a_k \\ & x_j = 0 \text{ or } 1, j \neq k \\ & s_i = 0, 1, 2, \dots, U_i. \end{aligned} \quad (17)$$

But the objective function in (17) is simply the left hand side of inequality (16). Thus, if the upper bound on $v(\lambda^0 | x_k = 1)$ is less than $\hat{v}(\lambda^0)$, the best known solution, the project k can be ignored without loss of optimality. ||

5. SENSITIVITY ANALYSIS OF EFFICIENT SOLUTIONS

For convex multicriterion optimization problems involving more than 2 criteria, it is generally not possible to describe completely the subsets $S(\lambda^k)$ of S in which x^k is an efficient solution (see (5)). Instead, the usual procedure is to find an efficient solution at λ^0 and see how it changes in the direction $\partial \lambda^0$ (Evans and Steuer [1973], Geoffrion et al [1972]). For the $\mathbb{P}$ multicriterion optimization problem (3), we are able to perform this sensitivity analysis through the dual problem.

Specifically, suppose we have constructed an $\mathbb{P}$ dual problem which solves (3), and let $(x^1, s^1) \in Y$ denote the optimal solution to (3) found by the $\mathbb{P}$ dual. We have

$$v(\lambda^0) = w(\lambda^0) = \lambda^0 C x^1$$

and moreover,

$$\begin{aligned} w(\lambda^0) &= \max \sum_{t=1}^T (\lambda^0 C x^t) \omega_t \\ \text{s.t. } & \sum_{t=1}^T (A x^t + I s^t) \omega_t = b \\ & \sum_{t=1}^T \omega_t = 1; \quad \omega_t \geq 0; \end{aligned} \quad (18)$$

that is, $\omega_1 = 1$, all other $\omega_t = 0$ is optimal in this linear programming problem. Let $w(\lambda)$ denote the maximal objective function value in (18) as λ varies. The functions $v(\lambda)$ and $w(\lambda)$ are convex functions satisfying $w(\lambda) \geq v(\lambda)$ by duality. In principle, we have only to perform linear programming sensitivity analysis on (18)

to compute the maximal value of $0 \leq \theta$ such that $\omega_1 = 1$ remains optimal in (18) with $\lambda = 1^0 + \theta \delta^0$ and therefore that (x^1, s^1) remains optimal in the IP multicriterion problem (3). The difficulty is that the number of columns in (18) will be enormous and column generation procedures are required. We omit further details except to state that generalized programming, otherwise known as Dantzig-Wolfe decomposition (see Lasdon [1970], Magnanti, Shapiro and Wagner [1973], can be used for this purpose.

6. CONCLUSIONS

In this paper, we have discussed multicriterion IP problems and how recent results in IP duality theory can be used to generate efficient solutions to these problems. The results presented here are far from complete and much research and practical experimentation remains to be done.

As we saw in Section 2, the IP dual methods are theoretically complete in the sense that an IP dual problem can always be constructed by a finite procedure which solves a given IP problem. Practical experience (Fisher, Northup and Shapiro [1974], however, has shown that the IP dual methods sometimes, but not always, require excessive numerical calculations. The natural resolution of this difficulty is to combine the dual methods with branch and bound (see Fisher and Shapiro [1974]). The procedure of section 5 needs to be reconciled with this practical consideration. In addition, the branch and bound procedures should be designed so that only one pass is made through the tree of enumerated solutions in computing the efficient x^k optimal in (3) as λ varies in int S . More generally, theoretical research is needed into the structure of the family of IP duals (14) generated as λ varies in S .

There are properties of efficient solutions to problem (1) which it is important to study further and try to characterize. For example, we would like to be able to identify the set of all projects j which are included in all efficient solutions. Conversely, we would like to identify the set of all projects excluded from all efficient solutions.

Another phenomenon to be studied which we have ignored here are the dynamics of public investment decision making. We eliminated an explicit study of dynamics by formulating the static model (1) of a T period problem.

As mentioned in the introduction, some public investment decision models involve distribution as well as location sub-problems implying the appropriateness of mixed IP multicriterion models. The IP duality theory can be combined with Benders' decomposition method for mixed IP (Shapiro [1974]). Thus, it appears that the approach suggested in this paper can be extended to these problems. Finally, we may wish to use a concave utility function u_p in the objective function of (1) that is, $\max (u_1(x), \dots, u_p(x))$ instead of $\max Cx$. The study of such problems is another

area of future research.

REFERENCES

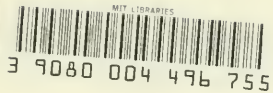
1. Bell, D.E., (1973a), "The Resolution of Duality Gaps in Discrete Optimization", Tech Report No. 81, Operations Research Center, Massachusetts Institute of Technology.
2. ——— (1973b), "Improved Bounds for Integer Programs: A Supergroup Approach," IASA RM-73-5, International Institute for Applied Systems Analysis, Laxenburg, Austria (to appear in SIAM J. on Applied Math).
3. ——— and J.F. Shapiro (1975), "A Finitely Convergent Duality Theory for Zero-One Integer Programming", Operations Research Center Working Paper No. OR 043-75, Massachusetts Institute of Technology.
4. Bharía, R., (1975), "Investment Planning for Petroleum and Fertilizer Industries: An Interregional Programming Model for India", Center for Population Studies Report, Harvard University.
5. Dorfman, R., and H.D. Jacoby (1972), "An Illustrative Model of a Regional Water Quality Authority", in R. Dorfman, H.D. Jacoby, and H. A. Thomas, Jr. (eds.), Models for Managing Regional Water Quality, Harvard University Press.
6. Evans, J.P. and R.E. Steuer (1973), "A Revised Simplex Method for Linear Multiple Objective Programs", Mathematical Programming, Vol. 5, 1, (August), 54-72.
7. Fisher, M.L., and J.F. Shapiro (1974), "Constructive Duality in Integer Programming", SIAM Journal on Applied Mathematics, 27, 31-52.
8. ———, W.D. Northup and J.F. Shapiro (1974), "Using Duality to Solve Discrete Optimization Problems: Theory and Computational Experience", Operations Research Center Working Paper No. OR 030-74, Massachusetts Institute of Technology (to appear in Mathematical Programming).
9. Geoffrion, A.M., J.S. Dyer and A. Peinberg, (1972), "An Interactive Approach for Multi-Criterion Optimization, with an Application to the Operation of an Academic Department", Management Science, Vol. 19, pp. 357-368.
10. Glover, F., (1969), "Integer Programming over a Finite Additive Group", SIAM Journal on Control 7, 213-231.
11. Goreux, L.M., and A. S. Manne (1973), Multi-level Planning: Case Studies in Mexico, North-Holland.
12. Gorry G.A., W.D. Northup, and J.F. Shapiro (1973), "Computational Experience with a Group Theoretic Integer Programming Algorithm", Mathematical Programming, 4, 171-192.
13. Karlin, S. (1959), Mathematical Methods and Theory in Games, Programming and Economics, Volume 1, Addison-Wesley,
14. Kendrick, D., and A. Stoutjesdijk (1975), The Planning of Industrial Investment Programs: A Methodology, (Manuscript in progress).

15. Kuhn, H.W., and A. W. Tucker (1950), "Nonlinear Programming", Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, California, pp. 481-492.
16. Kuhner, J., and J.J. Harrington (1974), "Large-Scale Mixed Integer Programming for Investigating Multi-Parity Public Investment Decisions: Application to a Regional Solid Waste Management Problem", (presented at 45th National ORSA/TMS Meeting, Boston).
17. Magnanti, T.L., J.F. Shapiro and M.H. Wagner (1973), "Generalized Linear Programming Solves the Dual", Operations Research Center Working Paper, No. 019-73, Massachusetts Institute of Technology, (to appear in Management Science).
18. Lasdon, L.S., (1970), Optimization Theory for Large Systems, Macmillan.
19. Marks, D.H. (1974), "Modelling for Policy Analysis in a Large Scale Underdeveloped River Basin", (presented at ORSA/TMS Meeting, April, 1974, Boston).
20. Shapiro, J.F. (1968), "Dynamic Programming Algorithms for the Integer Programming Problem - I: The Integer Programming Problem Viewed as a Knapsack Type Problem", Operations Research, 16, 103-121.
21. _____ (1971), "Generalized Lagrange Multipliers in Integer Programming", Operations Research, 19, 68-76.
22. _____ (1974), "A New Duality Theory for Mixed Integer Programming", Operations Research Center Working Paper No. OR 033-74, Massachusetts Institute of Technology (to appear in the Proceedings of the Congress on Mathematical Programming and Its Applications, Rome, April, 1974).
23. Westphal, L.E. (1971), Planning Investments with Economies of Scale, North-Holland

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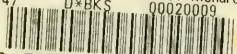
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724747 D*BKS 00020009



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724738 D*BKS 00020655



3 9080 000 656 295

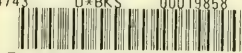
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T-J5-143-w no.786-75a
Kobrin, Stephe/Large scale direct OPEC
726385 D*BKS 00020654



3 9080 000 656 27

HD28.M414 no.787-75
Merton, Robert/Option pricing when und
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Plovnick, Mark/Issues in the design of
724742 D*BKS 00185829



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